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### Optimal Parametric Auctions

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## Abstract

We study the problem of an auctioneer who wants to maximize her profits. In our model, there are  $n$  buyers with private valuations drawn from independent distributions  $F_1, \dots, F_n$ . When these distributions are known to the seller, Myerson's optimal auction [20] is a well known mechanism that maximizes revenue. However, in many cases it is too strong to assume that the seller knows these distributions.

We propose an alternative model where the seller only knows the mean  $\mu_i$  and variance  $\sigma_i^2$  of each distribution  $F_i$ . We call mechanisms that only use this information *parametric auctions*. We construct such auctions for settings where the seller only has one copy of the good to sell, and when she has an infinite number of identical copies of the good (digital auctions). For a very large class of distributions, including (but not limited to) distributions with a monotone hazard rate, our auctions achieve a constant fraction of the revenue of Myerson's auction.

When the seller has absolutely no knowledge about the distributions, it is well known that no auction can achieve a constant fraction of the optimal revenue when the players are not identically distributed. Our parametric model gives the seller a small amount of extra information, allowing her to construct auctions for which (1) she does not know the full distribution of valuations, (2) no two bidders need to be drawn from identical distributions and (3) the revenue obtained is a constant fraction of the revenue in Myerson's optimal auction.

In addition to being competitive with traditional benchmarks, our digital parametric auction is optimal in a new sense, which we call *maximin optimality*. Informally, an auction is maximin optimal if it maximizes revenue in the worst case over an adversary's choice of the distribution. We show that our digital parametric is maximin optimal among the class of posted price mechanisms.

## 1 Introduction

We study the problem of selling a good in an auction, when the seller has limited information about what buyers are willing to pay. Specifically, we consider the problem of selling a good to  $n$  buyers, where each buyer is interested in purchasing one unit of the good. Each player  $i$  has a value  $v_i$  for the good, drawn from a distribution  $F_i$ . Starting with the work of Myerson [20], one traditionally assumes that the seller knows these distributions  $F_1, \dots, F_n$ . This assumption allows the seller to design an auction that maximizes her revenue.

**Parametric Auctions.** The assumption that the seller has full knowledge of the distributions  $F_1, \dots, F_n$  is very strong. We use a strictly weaker assumption, namely

*the seller only knows the mean  $\mu_i$  and standard deviation  $\sigma_i$  of each  $F_i$ .*

We call auctions where the seller only needs to know these parameters of the distribution *parametric auctions*. We construct such auctions both for *single good settings* and for *digital good settings*, and show that they obtain a significant fraction of the revenue obtainable by a seller who has full knowledge of the distributions  $F_1, \dots, F_n$ .

**Our Benchmark.** Letting  $\mathbf{F} = F_1 \times \dots \times F_n$  be the distribution from which valuations are drawn, our benchmark is the revenue obtained by Myerson's optimal auction on  $\mathbf{F}$ . We denote this benchmark by  $Rev(OPT, \mathbf{F})$ . More generally, for any auction  $A$  and any distribution  $\mathbf{F}$  we denote by  $Rev(A, \mathbf{F})$  the expected revenue that  $A$  obtains when valuations are drawn from  $\mathbf{F}$ , and by  $Rev_i(A, \mathbf{F})$  the expected revenue that auction  $A$  obtains from player  $i$ . Thus,  $Rev(A, \mathbf{F}) = \sum_{i=1}^n Rev_i(A, \mathbf{F})$ . For every auction  $A$  that we construct, we will seek to give a lower bound on the *competitive ratio*

$$\frac{Rev(A, \mathbf{F})}{Rev(OPT, \mathbf{F})}.$$

The idea of evaluating auctions by analyzing their competitive ratio was introduced by [16], and has been used widely in the literature. An incomplete list of applications includes prior-free auctions [14, 15], prior-independent auctions [10], simple<sup>1</sup> auctions [18], auctions with correlated bidders (where the optimal auction is not known) [23, 11, 21] and auctions where bidders have multi-dimensional types [6].

## 1.1 Performance relative to the Optimal Auction

**Our Results for Digital Goods with Monotone Hazard Rate Distributions.** We construct a parametric auction  $\mathcal{A}$  for the digital goods setting. When each distribution  $F_i$  has a monotone hazard rate, we can show that

$$\frac{Rev(\mathcal{A}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq \frac{1}{e} > 36\%.$$

We remark that our results hold no matter how many players there are, and what asymmetries may exist among them. (Notice that the best prior-free auctions can only obtain a constant fraction of Myerson's revenue when players are identically and independently distributed.)

**Our Results for Digital Goods with Arbitrary Distributions.** Our auction  $\mathcal{A}$  guarantees good revenue only when each distribution  $F_i$  has a monotone hazard rate. For arbitrary independent distributions, we construct a different digital auction  $\mathbb{A}$ . The competitive ratio of  $\mathbb{A}$  will be constant in the number of players and will not depend on the asymmetries of the players. However, it will be a function of the ratios  $\frac{\mu_i}{\sigma_i}$ . This is intuitive because if  $\sigma_i$  is very large compared with  $\mu_i$ , then the seller has very little information about  $F_i$ , and one cannot expect her to collect significant revenue from player  $i$ .

More concretely, we can give the following bound on the expected revenue that our auction  $\mathbb{A}$  obtains from each player  $i$

$$\frac{Rev_i(\mathbb{A}, \mathbf{F})}{Rev_i(OPT, \mathbf{F})} \geq \rho\left(\frac{\mu_i}{\sigma_i}\right),$$

where  $\rho(\cdot)$  is a function of  $\frac{\mu_i}{\sigma_i}$  that we define in section 4. We remark that

- $\rho(\frac{\mu_i}{\sigma_i})$  is always positive.
- When each  $F_i$  has a monotone increasing hazard rate,  $\rho(\frac{\mu_i}{\sigma_i}) > 10.5\%$ .
- More generally, for a very wide class of distributions  $\mathcal{F}_c$ , the function  $\rho(\frac{\mu_i}{\sigma_i})$  is bounded below by a constant. Let  $c$  be a constant and define  $\mathcal{F}_c = \{F : \mathbb{E}[F] = \mu, Var(F) = \sigma^2, \frac{\mu}{\sigma} > c > 0\}$ . When  $F_i$  belongs to this class, we have  $\rho(\frac{\mu_i}{\sigma_i}) > \rho(c) > 0$ . We remark that the class  $\mathcal{F}_c$  is very general. Any distribution  $F$  over a bounded domain  $[0, B]$  with mean  $\mu > cB$  is in  $\mathcal{F}_c$ . Furthermore, any distribution with a monotone hazard rate belongs to the class  $\mathcal{F}_1$ .

This last observation implies that for all  $F_1, \dots, F_n$  belonging to the class  $\mathcal{F}_c$ , our auction  $\mathbb{A}$  has a competitive ratio

$$\frac{Rev(\mathbb{A}, \mathbf{F})}{Rev(OPT, \mathbf{F})} > \rho(c).$$

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<sup>1</sup>The authors study, among others, Vickrey auctions with reserve prices, which are much simpler than the Myerson optimal auction when distributions are not identical.

**Our results for Single Good Auctions with Monotone Hazard Rate Distributions.** We construct a parametric auction  $\mathcal{B}$  for single-good settings. When each distribution  $F_i$  has a monotone hazard rate, our auction  $\mathcal{B}$  obtains a competitive ratio

$$\frac{Rev(\mathcal{B}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq 1.23\%.$$

**Our results for Single Good Auctions with Regular Distributions.** More generally, when the distributions  $F_1, \dots, F_n$  are regular<sup>2</sup> and belong to the class  $\mathcal{F}_c$  for some constant  $c$ , our auction  $\mathcal{B}$  has a competitive ratio of

$$\frac{Rev(\mathcal{B}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq \frac{1}{2} \cdot \psi(c) > 0$$

where  $\psi(x)$  is a positive increasing function defined in section 7.

We remark that, when the distributions  $F_1, \dots, F_n$  are unknown to the seller, most of the known auctions with a constant competitive ratio are digital goods auctions. There are few single-item competitive auction that we are aware of, and all of them assume some similarity between bidders.<sup>3</sup> In contrast, our auction  $\mathcal{B}$  obtains a constant competitive ratio even when all buyers have distinct distributions.

## 1.2 Maximin Optimality and our digital auction $\mathbb{A}$

Our digital auction  $\mathbb{A}$  is competitive with the optimal auction, where the competitive ratio depends on  $\frac{\mu_i}{\sigma_i}$ . Examining the competitive ratio is a meaningful way to give revenue guarantees when the full distribution is not known. However, there can be a multiplicity of different auctions which achieve a constant competitive ratio, and it may be difficult to decide which one is the “best” among all parametric auctions.

We restrict ourselves to a simple class of digital auctions, called *posted price mechanisms*. In these mechanisms, each player  $i$  is given a take-it-or-leave-it price that does not depend on the other players’ bids. We will show that our auction  $\mathbb{A}$  is one such mechanism.

Ideally, the “best” posted price mechanism  $A^*$  should satisfy  $Rev(A^*, \mathbf{F}) \geq Rev(A, \mathbf{F})$  for all distributions  $\mathbf{F}$  and all other posted price mechanisms  $A$ . Indeed, if the seller knows the distribution  $\mathbf{F}$ , an optimal posted price mechanism exists: it is Myerson’s optimal digital auction. Unfortunately, it is unlikely that such a mechanism exists in the parametric case.<sup>4</sup> Any definition of optimality for parametric auctions needs to take into account the uncertainty that the seller has over the distribution  $\mathbf{F}$ . Since this is a *worst-case uncertainty*, we give a new definition of optimality for parametric auctions which is based on maximizing the worst case revenue.

**Definition 1.** Let  $\mathcal{C}$  be a class of mechanisms. A parametric mechanism  $A^*$  is maximin optimal for the class  $\mathcal{C}$  if  $A^* \in \mathcal{C}$  and for all vectors  $\boldsymbol{\mu}, \boldsymbol{\sigma}$  we have

$$A^* \in \operatorname{argmax}_{A \in \mathcal{C}} \min_{\mathbf{F}: \mathbb{E}[\mathbf{F}] = \boldsymbol{\mu}, \operatorname{Var}(\mathbf{F}) = \boldsymbol{\sigma}^2} Rev(A, \mathbf{F}).$$

We show in section 6 that our digital auction  $\mathbb{A}$  is maximin optimal for the class of posted price mechanisms.

<sup>2</sup>A distribution is regular if its associated virtual valuation function is increasing.

<sup>3</sup>For example, Dhangwatnotai, Roughgarden, and Yan [10] assume that for any bidder  $i$  with a valuation  $v_i$  drawn from a distribution  $F$ , there exist another bidder  $j$  whose valuation  $v_j$  is also drawn from the same distribution  $F$ . See the related work section for more details.

<sup>4</sup>Informally, consider any price vector  $(p_1, \dots, p_n)$  and a posted price mechanism  $A$  that uses  $p_1, \dots, p_n$  as reserve prices. If the  $p_1, \dots, p_n$  are within a certain range (that depends on the given means and standard deviations), there will exist a distribution  $\mathbf{F}$  on which auction  $A$  is optimal. However, for a different vector  $(p'_1, \dots, p'_n)$  in the same range, there will exist a different distribution  $\mathbf{F}'$  on which a different auction  $A'$  will be optimal.

### 1.3 Our Techniques

We make extensive use of Chebyshev-type inequalities. These bounds are frequently used in robust optimization, but to the best of our knowledge they have not been applied to auction theory or mechanism design before. While we focus on revenue maximizing single item and digital auctions in this paper, we believe that these techniques can be more widely applied in other areas of mechanism design.

## 2 Preliminaries

**Bayesian Valuations.** We assume that valuations are drawn from a probability distribution. The  $i^{th}$  buyer's valuation is a random variable  $V_i$  over some domain  $D_i \subset \mathbb{R}_+$ .  $V_i$ 's cumulative distribution function is  $F_i : D_i \rightarrow [0, 1]$ , where  $F_i(x) = \Pr[V_i \leq x]$ . We denote by  $\mathbf{V} = (V_1, \dots, V_n)$  the vector of valuations, by  $D = D_1 \times \dots \times D_n$  its domain, and by  $\mathbf{F} = F_1 \times \dots \times F_n$  its joint distribution function. We denote by  $\mathbf{v} = (v_1, \dots, v_n) \in D$  the vector of realized valuations. When we want to emphasize player  $i$ 's valuation, we write the vector  $\mathbf{v} = (v_1, \dots, v_n)$  as  $\mathbf{v} = (v_i, v_{-i})$ .

**Auctions** An auction is given by a pair  $(A, P)$  where  $A : D \times \Delta D \rightarrow [0, 1]^n$  is an allocation rule and  $P : D \times \Delta D \rightarrow \mathbb{R}_+^n$  is a payment rule. If the auctioneer faces a bid vector  $\mathbf{v} = (v_1, \dots, v_n)$ , then he sells to player  $i$  with probability  $A_i(\mathbf{v})$ , and charges her a price  $P_i(\mathbf{v})$  when the item is sold. Each player can only be sold one copy of the good.

We emphasize that  $A(\mathbf{v}, \mathbf{F})$  not only depends on the valuations  $v_1, \dots, v_n$  but also on the distribution  $\mathbf{F}$ , but will sometimes write  $A_i(\mathbf{v}, \mathbf{F}) = A_i(\mathbf{v})$  when  $\mathbf{F}$  is clear from context. Furthermore, we will often take  $v_{-i}$  as fixed and write  $A_i(\mathbf{v}, \mathbf{F}) = A_i(v_i)$ .

**Single Item Settings and Digital Goods Settings.** We study two types of problems. In the *single item setting*, the seller has only one good to sell. We formalize this restriction by imposing the constraint  $\sum_{i=1}^n A_i(\mathbf{v}, \mathbf{F}) \leq 1$  on our auctions. In the *digital goods setting*, the seller has infinitely many copies of the good to sell. In this case there are no restrictions on the allocation function  $A$ , as long as each player can be sold a copy of the good with probability at most 1.

**Truthfulness and Monotonicity** If player  $i$  obtains the good with probability  $A_i$  and pays a price  $P_i$ , her utility is  $v_i \cdot A_i - P_i$ . A buyer with valuation  $v_i$  can attempt to increase her utility by lying, and reporting a bid  $v'_i \neq v_i$ . An auction is *truthful* if players have no incentive to misreport their true valuation. That is, for every player  $i$ , for every valuation vector  $\mathbf{v}$ , and every  $v'_i \neq v_i$ , we have  $v_i \cdot A_i(\mathbf{v}) - P_i(\mathbf{v}) \geq v_i \cdot A_i(\mathbf{v}') - P_i(\mathbf{v}')$ , where  $\mathbf{v}' = (v'_i, v_{-i})$ . It is well known [20, 1] that the auction  $(A, P)$  is truthful if and only if  $A_i(v_i, v_{-i})$  is monotonic in  $v_i$  and  $P_i(v_i, v_{-i}) = A_i(v_i, v_{-i})v_i - \int_0^{v_i} A_i(z_i; v_{-i})dz_i$ . An important corollary is that, for any monotonic allocation rule  $A(\cdot)$ , there exists a unique payment rule  $P(\cdot)$  that makes the auction  $(A, P)$  truthful. Thus, it suffices to specify a monotonic allocation rule  $A(\cdot)$  to specify a truthful auction.

**Deterministic Auctions.** The value  $A_i(\mathbf{v})$  is the probability that player  $i$  obtains the good given that the bid vector is  $\mathbf{v}$ . We focus on deterministic allocations, where  $A_i(\mathbf{v}) \in \{0, 1\}$ . If an allocation  $A_i(\mathbf{v})$  is deterministic and truthful, then monotonicity implies that, for every  $v_{-i}$ , there exists a reserve price  $p^*(v_{-i})$  such that the auction sells to player  $i$  when  $v_i > p^*(v_{-i})$ . The payment that makes this allocation truthful is charging player  $i$  a price of  $p^*(v_{-i})$  dollars if she wins.

**Monotone Hazard Rate and Regularity** Given a differentiable cumulative distribution function  $F_i$ , let  $f_i(v) = \frac{d}{dv} F_i(v)$  be its induced density function. The function  $h_i(v) = \frac{f_i(v)}{1-F_i(v)}$  is called the *hazard rate* of  $F_i$ . The distribution  $F_i$  has a *monotone hazard rate* if  $h_i(v)$  is increasing. The distribution  $F_i$  is called *regular* if the virtual valuation function  $\phi_i(v) = v - \frac{1}{h_i(v)}$  is increasing. An immediate consequence is that any distribution with a monotone hazard rate is regular.

**Revenue** We denote by  $Rev((A, P), \mathbf{F}) = \int_D \sum_{i=1}^n A_i(\mathbf{v}) P_i(\mathbf{v}) d\mathbf{F}(\mathbf{v})$  the expected revenue obtained by an auction  $(A, P)$  when valuations follow distribution  $F$ , and by  $Rev_i((A, P), \mathbf{F}) = \int_D A_i(\mathbf{v}) P_i(\mathbf{v}) d\mathbf{F}(\mathbf{v})$  the expected revenue obtained from player  $i$ .

**Distribution Parameters.** Player  $i$ 's valuation is a random variable  $V_i$  with mean  $E[V_i] = \mu_i$ ,  $E[(V_i - \mu_i)^2] = \sigma_i^2$ . We will write  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$  and  $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ .

**Parametric Auctions** Informally, an auction  $(A, P)$  is *parametric* if its allocation and payment functions can be computed from the valuation vector  $\mathbf{v}$  and the  $\boldsymbol{\mu}, \boldsymbol{\sigma}$  parameters. More formally, we have the following definition.

**Definition 2.** A *parametric auction* is a pair of functions  $(A, P)$

1.  $A : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow [0, 1]^n$
2.  $P : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$
3.  $A_i(\mathbf{v}, \boldsymbol{\mu}, \boldsymbol{\sigma})$  is the probability that player  $i$  wins a copy of the good when bids are  $\mathbf{v}$  and the mean and standard deviation vectors of the distribution are  $\boldsymbol{\mu}, \boldsymbol{\sigma}$ .
4.  $P_i(\mathbf{v}, \boldsymbol{\mu}, \boldsymbol{\sigma})$  is the price that player  $i$  has to pay when bids are  $\mathbf{v}$  and the mean and standard deviation vectors of the distribution are  $\boldsymbol{\mu}, \boldsymbol{\sigma}$ .

For convenience of notation, whenever  $\boldsymbol{\mu}$  and  $\boldsymbol{\sigma}$  are clear we will write  $A(\mathbf{v})$  and  $P(\mathbf{v})$ .

**Posted Price Mechanisms** A posted price mechanism for digital goods is a digital auction where player  $i$  is offered a take-it-or-leave-it price  $p_i(\mathbf{F})$  that does not depend on the players' bids. Player  $i$  gets a copy of the good if and only if  $v_i > P_i(\mathbf{F})$ . The optimal digital auction when  $\mathbf{F}$  is known is a posted price mechanism. Player  $i$  is given a price  $p_i^* = \operatorname{argmax}_{p_i} p_i \cdot (1 - F_i(p_i))$ . All of the parametric digital auctions that we construct are posted price mechanisms, where  $P_i$  only depends on the parameters  $\mu_i, \sigma_i$  of player  $i$ 's distribution.

## 3 Further Related Work

### 3.1 Detail-Free Mechanisms

The Wilson doctrine [26] states that a good mechanism should require as little knowledge about the valuations of the players as possible. Our paper follows the spirit of the Wilson doctrine, removing the assumption that the seller knows the distribution of buyer valuations and replacing it by the strictly weaker assumption that the seller knows only the first and second moments of these distributions.

Baliga and Vohra [2] and Segal [24] have proposed explicit detail-free bayesian auctions where the seller does not need to know the distribution of valuations. However, these auctions are competitive with the optimal auction only when the buyers' valuations are identically and independently distributed from the same (unknown) distribution  $F$ .

**Prior-Free auctions.** Goldberg, Hartline, Karlin, Saks and Wright [16] consider auctions where the valuations are not necessarily drawn from a distribution. To measure the performance of the auctions, they introduce a revenue benchmark  $\mathcal{F}_2(v_1, \dots, v_n)$ ,<sup>5</sup> and construct auctions that always achieve a constant fraction of this benchmark, for all valuation vectors. As made explicit by Hartline and Roughgarden [17], any auction that achieves a constant fraction of  $\mathcal{F}_2$  on all valuation vectors, will also achieve a constant fraction of the optimal auction's revenue, as long as valuations are identically and independently distributed.

<sup>5</sup>Do not confuse the revenue benchmark  $\mathcal{F}_2$  with our distribution classes  $\mathcal{F}_c$ . In our results,  $\mathcal{F}_c$  will always refer to a class of distributions.

Our results are incomparable to the existing prior-free auctions. We assume that the valuations are drawn from a distribution, and that the seller has some limited information about this distribution. However, all the previous prior-free auctions are competitive with the optimal auction only when the valuations are independently and identically distributed. In contrast, our results apply in the more general setting where buyers' valuations can be drawn from distinct distributions.

**Prior Independent Auctions** An auction is prior-independent if the distributions  $F_1, \dots, F_n$  are assumed to exist, but the seller does not know what they are. In this setting, Dhangwatnotai, Roughgarden and Yan [10] construct an auction that has a constant competitive ratio, as long as bidders satisfy a symmetry condition which they call non-singularity. Their analysis requires that, for every bidder  $i$ , there exist another bidder  $j$  whose valuation is drawn from the same distribution as  $i$ 's valuation. Our result is again, incomparable to theirs. By assuming that the seller knows the mean and variance of each  $F_i$ , we can give approximately optimal auctions even when the bidders are *arbitrarily asymmetric*. That is, our auctions do not require any two distributions to be identical in order to guarantee a constant fraction of Myerson's revenue.

We are aware that the mechanisms in [10] can be transformed into competitive auctions with asymmetric bidders where the auctioneer gets one sample from each distribution  $F_1, \dots, F_n$ . These samples are "extra information", in the same way that the mean and variance of the distributions are extra information. We remark that our assumptions are incomparable: the seller may know  $\mu_i$  and  $\sigma_i$  without having access to samples from the distribution  $F_i$ , and knowing one sample does not give a good estimate of the mean or variance.

### 3.2 Empirical Estimation of Auctions

One of the main motivations of our work is the application of optimal auctions in practice. This requires estimating the distribution of bidder values. Ostrovsky and Schwarz [22] study the effect of reserve prices on ad auctions. They derive their reserve prices by estimating means and standard deviations of bidder values using previous auction data, and then assuming that valuations are drawn from a log-normal distribution with the estimated mean and standard deviation.

Our work proposes a more conservative way to set reserve prices. Instead of finding these prices by assuming that valuations are drawn from a specific distribution family (in the above case, a log-normal distribution), our auction sets the lowest reserve price that is compatible with any distribution with mean  $\mu$  and standard deviation  $\sigma$ . We are very interested in doing an empirical analysis of auctions using this more conservative way of setting reserve prices.

### 3.3 The maximin approach to robust mechanism design

We defined a maximin optimal mechanism as one that maximizes revenue in the worst case over the choice of the distribution  $\mathbf{F}$ , as long as  $\mathbf{F}$  has the pre-specified mean  $\mu$  and standard deviation  $\sigma$ .

This is a new definition, but it is not without precedent. It follows Wald's maximin model for decision making under non-bayesian uncertainty [25]. In this model, a decision maker has to maximize a function  $f(a, s)$ , that depends on her action  $a$  and an unknown state of the world  $s$ . Wald's model suggests that the player take an action  $a^* = \max_a \min_s f(a, s)$  that maximizes the worst case payoff over all possible states of the world.

As an example of this concept in mechanism design, Chung and Ely [8] study the problem of an auctioneer who knows the distribution of player valuations, but where the players can have arbitrary beliefs about each other. They show that a dominant strategy truthful auction will guarantee the maximum "worst-case revenue" in equilibrium, where the worst case is taken over the choice of players' beliefs.

## 4 Digital Parametric Auctions with Arbitrary Independent Distributions

We now construct a digital auction  $\mathbb{A}$  which is competitive with the optimal auction. The competitive ratio will depend on the ratio  $\frac{\mu_i}{\sigma_i}$ .

Our parametric digital auction  $\mathbb{A}$

- $\mathbb{A}(\mathbf{v}, \boldsymbol{\mu}, \boldsymbol{\sigma})$
- 1 Find  $k_i = \operatorname{argmax}_t [(\mu_i - \sigma_i t) \cdot \frac{t^2}{1+t^2}]$ .
  - 2 For each player  $i$ , set the reserve price  $r_i = \mu_i - \sigma_i k_i$ .
  - 3 Sell a copy of the good to player  $i$  if and only if  $v_i > r_i$ .

**Theorem 1.** *For any distribution  $\mathbf{F}$  with mean  $\boldsymbol{\mu}$  and standard deviation  $\boldsymbol{\sigma}$ , we have*

$$\frac{\operatorname{Rev}_i(\mathbb{A}, F)}{\operatorname{Rev}_i(\operatorname{OPT}, F)} \geq (1 - \frac{3}{2} \frac{\sigma}{\mu} k_i) = (\frac{1}{2} \frac{\sigma}{\mu} k_i^3).$$

*Proof.*

We prove this via a series of lemmas. First, we characterize  $k_i$  in terms of  $\frac{\mu_i}{\sigma_i}$ .

**Lemma 1.** *Let  $r_i = \mu_i - \sigma_i k_i$  be player  $i$ 's reserve price in auction  $\mathbb{A}$ . We have that  $k_i$  is the unique real solution to the cubic equation  $\frac{\mu_i}{\sigma_i} = \frac{1}{2}(3k_i + k_i^3)$*

*Proof of Lemma 1.* The value  $k_i$  is obtained by maximizing the differentiable function  $(\mu_i - \sigma_i k) \cdot \frac{k^2}{1+k^2}$  over  $k \geq 0$ . Note that finding  $k_i$  is equivalent to finding the value  $k$  maximizing  $\ln(\mu_i - \sigma_i k) + 2 \ln k - \ln(1 + k^2)$ . Taking derivatives of this function, we obtain that  $k_i$  satisfies the equation

$$-\frac{\sigma_i}{\mu_i - \sigma_i k_i} + \frac{2}{k_i} - \frac{2k_i}{1 + k_i^2} = 0.$$

Multiplying the denominators out, we get

$$-\sigma_i k_i (1 + k_i^2) + 2(1 + k_i^2)(\mu_i - \sigma_i k_i) - 2k_i^2(\mu_i - \sigma_i k_i) = 0.$$

Canceling out some terms and rearranging gives

$$-\sigma_i k_i (1 + k_i^2) + 2(\mu_i - \sigma_i k_i) = 0$$

$$2\mu_i = 3\sigma_i k_i + \sigma_i k_i^3$$

$$\frac{\mu_i}{\sigma_i} = \frac{1}{2}(3k_i + k_i^3),$$

which is what we wanted to show.

Note that this cubic equation has exactly one real root because the function  $\frac{1}{2}(3k_i + k_i^3) - \frac{\mu_i}{\sigma_i}$  is strictly increasing. Indeed, its derivative is  $\frac{3}{2} + \frac{3}{2}k_i^2 > 0$ . This completes the proof of Lemma 1.  $\blacksquare$

With this characterization of  $k_i$  in hand, we can give a lower bound on the expected payment from player  $i$ .

**Lemma 2.** *Let  $\mathbf{F}$  be a distribution with mean  $\boldsymbol{\mu}$  and standard deviation  $\boldsymbol{\sigma}$ . The expected revenue that auction  $\mathbb{A}$  obtains from player  $i$  is at least  $\mu_i - \frac{3}{2}\sigma_i k_i = \frac{1}{2}\sigma_i k_i^3$ .*



*Proof of Lemma 2.* The expected revenue obtained from player  $i$  is  $(\mu_i - \sigma_i k_i) \cdot (1 - F_i(\mu_i - \sigma_i k_i))$ . However, we do not know the value  $1 - F_i(\mu_i - \sigma_i k_i)$ . We need to give a lower bound. To do this, we use the following one-sided version of Chebyshev's inequality.

(*Cantelli's Inequality*) For every real-valued distribution with mean  $\mu$  and variance  $\sigma^2$ , we have

$$1 - F(\mu - \sigma k) \geq \frac{k^2}{1 + k^2}$$

From Cantelli's inequality, we obtain a bound of  $(\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1 + k_i^2}$  on the revenue collected from player  $i$ . From Lemma 1, we know that  $k_i$  satisfies  $\frac{\mu_i}{\sigma_i} = \frac{1}{2}(3k_i + k_i^3)$ . Multiply both sides of the equation by  $\sigma_i$  to obtain  $\mu_i = \frac{1}{2}\sigma_i(3k_i + k_i^3)$ . Now we can write  $\mu_i - \sigma_i k_i = \frac{1}{2}\sigma_i(k_i + k_i^3) = \frac{1}{2}\sigma_i k_i(1 + k_i^2)$ . The lower bound on the expected auction revenue becomes

$$(\mu_i - \sigma_i k_i) \frac{k_i^2}{1 + k_i^2} = \frac{1}{2}\sigma_i k_i(1 + k_i^2) \frac{k_i^2}{1 + k_i^2} = \frac{1}{2}\sigma_i k_i^3.$$

Using again the fact that  $\mu_i - \sigma_i k_i = \frac{1}{2}\sigma_i(k_i + k_i^3)$ , we can write this revenue bound as  $\mu_i - \frac{3}{2}\sigma_i k_i$ . This completes the proof of Lemma 2. ■

We remark that if  $\sigma_i = 0$ , then the player's valuation is  $\mu_i$  with probability 1. Thus, the expected revenue from player  $i$  is  $\mu_i$ .

We have given a lower bound on the revenue that  $\mathbb{A}$  obtains from each player  $i$ . We can also give an upper bound on the revenue that the optimal auction obtains from player  $i$  by noting that a truthful, individually rational auction<sup>6</sup> will always charge a price  $p_i$  lower than the player's valuation  $v_i$ . Thus, the expected revenue of the auction satisfies  $\mathbb{E}[p_i] \leq \mathbb{E}[v_i] = \mu_i$ . Using this together with the above lemma, we can bound the player- $i$  competitive ratio of auction  $\mathbb{A}$  as follows

$$\frac{Rev_i(\mathbb{A}, \mathbf{F})}{Rev_i(OPT, \mathbf{F})} \geq \frac{\mu_i - \frac{3}{2}\sigma_i k_i}{\mu_i} = 1 - \frac{3}{2} \frac{\sigma_i}{\mu_i} k_i.$$

This completes the proof of the theorem. *Q.E.D.*

## 4.1 A constant competitive Ratio

Theorem 1 tells us that  $Rev_i(\mathbb{A}, \mathbf{F}) \geq (1 - \frac{3}{2} \frac{\sigma_i}{\mu_i} k_i) Rev_i(OPT, \mathbf{F})$  for every player  $i$ . Thus, we can bound the revenue over all players as

$$\sum_{i=1}^n Rev_i(\mathbb{A}, \mathbf{F}) \geq \sum_{i=1}^n (1 - \frac{3}{2} \frac{\sigma_i}{\mu_i} k_i) Rev_i(OPT, \mathbf{F}) \geq \min_i (1 - \frac{3}{2} \frac{\sigma_i}{\mu_i} k_i) \sum_{i=1}^n Rev_i(OPT, \mathbf{F}).$$

This gives us the competitive ratio

$$\frac{Rev(\mathbb{A}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq \min_i (1 - \frac{3}{2} \frac{\sigma_i}{\mu_i} k_i).$$

One key informal observation is that the competitive ratio  $\min_i (1 - \frac{3}{2} \frac{\sigma_i}{\mu_i} k_i)$  is constant whenever  $\frac{\mu_i}{\sigma_i}$  is bounded below by a constant for all players. Thus, for a very large class of distributions, our parametric auction obtains a constant fraction of the revenue. We remark that, unlike previous auctions, we make no assumptions on players having identical distributions.

More formally, make the following definitions:

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<sup>6</sup>An auction is individually rational if no player gets negative utility from participating. If a player does not buy a copy of the good, her price is zero. If she buys a copy of the good, her price  $p_i$  is less than her value  $v_i$ .

1. The function  $k(a)$  maps  $a$  to the unique real root of the cubic equation  $\frac{1}{2}(k^3 + 3k) = a$ .
2. The function  $\rho(a) \stackrel{\text{def}}{=} 1 - \frac{3}{2}\frac{1}{a}k(a)$ . We can also write  $\rho(a) = \frac{1}{2}\frac{1}{a}k^3(a)$  by our definition of  $k(a)$ . (See lemma 1).
3. For any constant  $c > 0$ , the class of distributions  $\mathcal{F}_c \stackrel{\text{def}}{=} \{F : \mathbb{E}[F] = \mu, \text{Var}(F) = \sigma^2, \frac{\mu}{\sigma} > c\}$ .

We can now show the following theorem.

**Theorem 2.** *For any constant  $c > 0$ , let  $\mathbf{F} = F_1 \times \dots \times F_n$  be a distribution where each  $F_i$  is in the class  $\mathcal{F}_c$ . Then*

$$\frac{\text{Rev}(\mathbb{A}, \mathbf{F})}{\text{Rev}(\text{OPT}, \mathbf{F})} \geq \rho(c).$$

*Proof.* Theorem 1 tells us that  $\frac{\text{Rev}(\mathbb{A}, \mathbf{F})}{\text{Rev}(\text{OPT}, \mathbf{F})} \geq \min_i \rho(\frac{\mu_i}{\sigma_i})$ . The fact that  $F_i \in \mathcal{F}_c$  implies that  $\frac{\mu_i}{\sigma_i} > c$ . If we show that the function  $\rho(\cdot)$  is increasing, we can conclude that  $\rho(\frac{\mu_i}{\sigma_i}) > \rho(c)$  for all  $i$ , which gives us

$$\min_i \rho(\frac{\mu_i}{\sigma_i}) > \rho(c).$$

Now we show that  $\rho(a)$  is an increasing function of  $a$  when  $a \geq 0$ . Since  $\rho(a) \stackrel{\text{def}}{=} 1 - \frac{3}{2}\frac{1}{a}k(a)$ , it's derivative is  $\rho'(a) = \frac{3}{2}\frac{1}{a^2}k(a) - \frac{3}{2}\frac{1}{a}k'(a)$ .

The function  $k(a)$  was defined implicitly as  $2a = k^3 + 3k$ . Implicit differentiation gives us  $2 = 3k^2 \cdot k'(a) + 3k'(a)$ , which we can rewrite as  $k'(a) = \frac{2}{3} \frac{1}{1+k^2(a)}$ . Plugging this into our expression for  $\rho'(a)$  we obtain

$$\rho'(a) = \frac{3}{2} \frac{1}{a^2} k(a) - \frac{1}{a} \cdot \frac{1}{1+k^2(a)}.$$

We can multiply the above equality by  $a^2 \cdot (1+k^2(a))$  without changing the sign of  $\rho'(a)$ . To prove  $\rho'(a) \geq 0$ , it suffices to show

$$\begin{aligned} a^2 \cdot (1+k^2(a))\rho'(a) &\geq 0 \\ \frac{3}{2}k(a)(k^2(a)+1) - a &\geq 0 \\ k^3(a) + \frac{1}{2}k^3(a) + \frac{3}{2}k(a) - a &\geq 0 \\ k^3(a) + 0 &\geq 0. \end{aligned}$$

Since  $k(0) = 0$  and  $k$  is an increasing function, we have  $k^3(a) \geq 0$  when  $a \geq 0$ . This shows that  $\rho(\cdot)$  is an increasing function when  $a > 0$ , and thus that  $\rho(\frac{\mu_i}{\sigma_i}) > \rho(c)$  when  $F_i \in \mathcal{F}_c$ . We can conclude that when all  $F_1, \dots, F_n \in \mathcal{F}_c$ , the competitive ratio of  $\mathbb{A}$  satisfies the bound

$$\frac{\text{Rev}(\mathbb{A}, \mathbf{F})}{\text{Rev}(\text{OPT}, \mathbf{F})} \geq \min_i \rho(\frac{\mu_i}{\sigma_i}) > \rho(c).$$

This completes the proof of the theorem. *Q.E.D.*

## 4.2 The competitive ratio of $\mathbb{A}$ when the distributions have a monotone hazard rate.

As mentioned in the introduction, the class  $\mathcal{F}_c$  is a very large class of distributions. In particular, any distribution  $F$  with a monotone hazard rate belongs to the class  $\mathcal{F}_1$ . Barlow, Marshall and Proschan [4] noted that for a random variable  $X$  with monotone hazard rate,  $E[X^2] \leq 2E[X]^2$ . Since  $\text{Var}(X) = E[X^2] - E[X]^2$ , this gives us  $\text{Var}(X) \leq E[X]^2$ . Taking square roots, we obtain  $\sigma \leq \mu$  or, equivalently,  $\frac{\mu}{\sigma} > 1$ .

Theorem 2 tells us that the auction  $\mathbb{A}$  has a competitive ratio greater than  $\rho(1)$  when the distributions  $F_1, \dots, F_n$  are in  $\mathcal{F}_1$ . In particular, this is true when the distributions have a monotone hazard rate. We can use this to conclude the following corollary.

**Corollary 1.** *When each  $F_i$  has a monotone hazard rate, the competitive ratio of  $\mathbb{A}$  is bounded below by*

$$\frac{Rev(\mathbb{A}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq \rho(1) > 10.5\%.$$

*Proof.* The fact that  $\frac{Rev(\mathbb{A}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq \rho(1)$  is immediate from Theorem 2 and the fact that a distribution with monotone hazard rate satisfies  $\frac{\mu}{\sigma} > 1$ . We need to show via a computation that  $\rho(1) > 10.5\%$ . Recall that  $\rho(1) = 1 - \frac{3}{2} \frac{1}{1} \cdot k(1)$ . Solving the cubic equation  $k^3 + 3k = 2$  gives us  $k(1) = \sqrt[3]{1 + \sqrt{2}} - \frac{1}{\sqrt[3]{1 + \sqrt{2}}} \cong 0.596$ . Plugging this into the formula for  $\rho(1)$ , we obtain  $\rho(1) > 10.5\%$ . ■

## 5 A Better Digital Auction for Distributions with Monotone Hazard Rate

In the previous section, we presented a digital auction  $\mathbb{A}$  and showed that it obtained a constant competitive ratio for a wide class of distribution, including all distributions with an increasing hazard rate.

In this section, we give a simpler auction  $\mathcal{A}$  which will guarantee an even better competitive ratio, under the assumption that each  $F_i$  has an increasing hazard rate. Unlike our auction  $\mathbb{A}$ , we cannot give any guarantees on the competitive ratio when the  $F_i$  do not have increasing hazard rates.

Our auction will use the following observation from Barlow and Marshall [3]:

**Fact 1.** *Let  $F$  be a distribution over the real numbers with monotone hazard rate. Let  $\mu$  be the mean of  $F$ . Then  $1 - F(\mu) \geq \frac{1}{e}$ .*

We can now describe our competitive parametric auction

$\mathcal{A}(v, \mu, \sigma)$

- 1 Set a reserve for player  $i$  at  $r_i = \mu_i$ .
- 2 Sell a copy of the good to player  $i$  if and only if  $v_i > r_i$ .

**Theorem 3.** *For all distributions  $\mathbf{F}$  with monotone hazard rate, we have*

$$\frac{Rev(\mathcal{A}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq \frac{1}{e}$$

*Proof.* The proof is immediate using Barlow and Marshall's observation and the fact that the optimal auction cannot obtain more than  $\mu_i$  expected revenue from player  $i$ .

The expected revenue that  $\mathcal{A}$  obtains from player  $i$  is  $\mu_i \cdot (1 - F(\mu_i))$ . By the above observation, this is greater than or equal to  $\mu_i \cdot \frac{1}{e}$ . Since the optimal auction cannot obtain more than  $\mu_i$  expected revenue from player  $i$ , our theorem is proved. *Q.E.D.*

## 6 Maximin Optimality and our auction $\mathbb{A}$

Recall our definition of maximin optimality for parametric auctions.

**Definition.** *Let  $\mathcal{C}$  be a class of mechanisms. A parametric auction  $A^*$  is maximin optimal for class  $\mathcal{C}$  if  $A^* \in \mathcal{C}$  and*

$$A^* \in \operatorname{argmax}_{A \in \mathcal{C}} \min_{\mathbf{F}: \mathbb{E}[\mathbf{F}] = \mu, \operatorname{Var}(\mathbf{F}) = \sigma^2} Rev(A, \mathbf{F}).$$

**Theorem 4.** *The auction  $\mathbb{A}$  is maximin optimal for the class  $\mathcal{C}$  of parametric posted price mechanisms.*

*Proof.* First, we note that  $\mathbb{A} \in \mathcal{C}$ , the class of posted price auctions. Player  $i$ 's posted price is  $r_i = \mu_i - \sigma_i k_i$ . Now we need to show that  $\mathbb{A}$  maximizes the worst-case revenue among any such mechanisms.

In Theorem 1, we used Cantelli's inequality to show that

$$\text{Rev}(\mathbb{A}, F) \geq \sum_{i=1}^n (\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1 + k_i^2}.$$

To prove Theorem 4, it suffices to show that for any  $\boldsymbol{\mu}, \boldsymbol{\sigma}$  and any parametric posted price mechanism  $A(\mathbf{v}, \boldsymbol{\mu}, \boldsymbol{\sigma})$ , there exists a distribution  $\mathbf{F}$  with mean  $\boldsymbol{\mu}$  and standard deviation  $\boldsymbol{\sigma}$  such that

$$\text{Rev}(A, \mathbf{F}) \leq \sum_{i=1}^n (\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1 + k_i^2}.$$

Since the digital auction  $A(\mathbf{v}, \boldsymbol{\mu}, \boldsymbol{\sigma})$  is a parametric posted price mechanism, it is completely characterized by a vector of reserve prices  $(p_1(\boldsymbol{\mu}, \boldsymbol{\sigma}), \dots, p_n(\boldsymbol{\mu}, \boldsymbol{\sigma}))$ . The auction sells a copy of the good to player  $i$  if and only if  $v_i > p_i$ . Thus, player  $i$ 's expected payment is  $p_i \cdot (1 - F_i(p_i))$ .

We will prove the theorem by showing that, for each player  $i$ , there exists a distribution  $F_i$  with mean  $\mu_i$  and variance  $\sigma_i$  on which player  $i$ 's expected payment is less than or equal to  $(\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1 + k_i^2}$ . We split into two cases:  $p_i > \mu_i$  and  $p_i \leq \mu_i$ .

Assume  $p_i > \mu_i$  and write  $p_i = \mu_i + \frac{\sigma_i}{t_i}$ , for some positive  $t_i$ . Consider the family of distributions  $\{C(k; \mu_i, \sigma_i)\}_{k \in \mathbb{R}}$  where  $C(k; \mu_i, \sigma_i)$  takes the values

$$H(k; \mu_i, \sigma_i) = \mu_i + \frac{\sigma_i}{k} \text{ with probability } \frac{k^2}{1 + k^2},$$

$$L(k; \mu_i, \sigma_i) = \mu_i - \sigma_i k \text{ with probability } \frac{1}{1 + k^2}.$$

Let player  $i$ 's valuation  $v_i$  be drawn from distribution  $C(k; \mu_i, \sigma_i)$ . When  $k$  is smaller than  $t_i$ , the auction will sell the good if and only if  $v_i = H(k; \mu_i, \sigma_i)$ , and it will charge player  $i$  the reserve price  $p_i = \mu_i + \frac{\sigma_i}{t_i}$ . The expected revenue collected from player  $i$  in this case is  $(\mu_i + \frac{\sigma_i}{t_i}) \cdot \frac{k^2}{1 + k^2}$ . Taking the limit as  $k \rightarrow 0$ , our expected revenue becomes arbitrarily small. In particular, it becomes smaller than  $(\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1 + k_i^2}$ .

Now assume  $p_i \leq \mu_i$ . Write  $p_i = \mu_i - \sigma_i t_i$ . Choose  $F_i$  to be the distribution  $C(t_i; \mu_i, \sigma_i)$ . For this distribution, the valuation  $v_i$  will be

$$H(t_i) = \mu_i + \frac{\sigma_i}{t_i} \text{ with probability } \frac{t_i^2}{1 + t_i^2},$$

$$L(t_i) = \mu_i - \sigma_i t_i \text{ with probability } \frac{1}{1 + t_i^2}.$$

The auction  $A$  only sells to player  $i$  when  $v_i > \mu_i - \sigma_i t_i$ .<sup>7</sup> For this particular distribution, the auction will sell at price  $\mu_i - \sigma_i t_i$ , but only when the valuation is  $H(t_i) = \mu_i + \frac{\sigma_i}{t_i}$ . This happens with probability  $\frac{t_i^2}{1 + t_i^2}$ . Thus, the expected revenue collected from player  $i$  will be

$$(\mu_i - \sigma_i t_i) \cdot \frac{t_i^2}{1 + t_i^2} \leq (\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1 + k_i^2},$$

where the last inequality is because  $k_i$  maximizes the function  $f(t) = (\mu_i - \sigma_i t) \cdot \frac{t^2}{1 + t^2}$ . We have shown that, when  $p_i < \mu_i$ , there exists a distribution  $F_i$  for which the expected revenue collected from player  $i$  is less than or equal to  $(\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1 + k_i^2}$ .

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<sup>7</sup>For this proof, the strictness of the inequality matters. If we sold when  $v_i \geq \mu_i - \sigma_i t_i$ , then a similar argument would apply, but we would need to use a distribution  $C(t_i + \epsilon; \mu_i, \sigma_i)$  for arbitrarily small values of  $\epsilon$ .

The above analysis holds for each player individually. However, since  $A$  is a digital auction, we can simply consider a product distribution  $\mathbf{F} = (F_1, \dots, F_n)$ , where each  $F_i$  is chosen to limit the amount of revenue that  $A$  collects from player  $i$ . Adding up over all players, we conclude that there exists a distribution  $\mathbf{F}$  such that

$$\text{Rev}(A, \mathbf{F}) \leq \sum_{i=1}^n (\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1 + k_i^2}.$$

Recalling that our parametric auction  $\mathbb{A}$  satisfies  $\text{Rev}(\mathbb{A}, F) \geq \sum_{i=1}^n (\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1 + k_i^2}$  for all distributions  $\mathbf{F}$  with mean  $\boldsymbol{\mu}$  and standard deviation  $\boldsymbol{\sigma}$ , we obtain that  $\mathbb{A}$  maximizes worst-case revenue. This concludes the proof of Theorem 4. *Q.E.D.*

## 7 Parametric Auctions For a Single Item

We present in this section a competitive parametric auction for selling a single item. When the distributions  $F_1, \dots, F_n$  are regular, our competitive ratio will be a function that depends on the distribution parameters. When the distributions  $F_1, \dots, F_n$  belong to the class  $\mathcal{F}_c$  for some constant  $c > 0$ , we will obtain a constant competitive ratio.

Our parametric auction will be a second-price auction with reserve prices  $\mathbf{r}$ . Because we do not know the distributions  $F_1, \dots, F_n$ , we will need to set the reserve price for player  $i$  using only the parameters  $\mu_i, \sigma_i$ . Recall the definition of a second-price auction with reserve prices  $\mathbf{r} = (r_1, \dots, r_n)$  [18].

$VCG_{\mathbf{r}}(\mathbf{v})$

- 1 Find the set  $S = \{i : v_i > r_i\}$  of players whose bids are above their reserve price.
- 2 Sell the good to the player  $i^* \in S$  with the highest valuation.
- 3 Let  $SP = \max_{j \neq i^*, j \in S} v_j$  be the second highest valuation among players in  $S$ . Charge player  $i^*$  either  $SP$  or her reserve price  $r_{i^*}$ , whichever is larger.

With this definition in hand, we can now define our single-good auction  $\mathcal{B}$ .

$\mathcal{B}(\mathbf{v}, \boldsymbol{\mu}, \boldsymbol{\sigma})$

- 1 Find  $k_i = \arg\max_t [(\mu_i - \sigma_i t) \cdot \frac{t^2}{1+t^2}]$ .
- 2 For each player  $i$ , set the reserve price  $q_i = \mu_i - \frac{3}{2}\sigma_i k_i$ .
- 3 Run the auction  $VCG_{\mathbf{q}}$  to allocate and price the good.

**Theorem 5.** *Let  $c > 0$  be a constant and let  $F_1, \dots, F_n$  be regular distributions in the class  $\mathcal{F}_c$ . There exists a positive increasing function  $\psi$  such that*

$$\frac{\text{Rev}(\mathcal{B}, \mathbf{F})}{\text{Rev}(\text{OPT}, \mathbf{F})} \geq \frac{1}{2}\psi(c) > 0$$

*Proof.* If the distribution  $F_i$  is known, one way to set the reserve price  $r_i$  is to choose  $r_i = p_i \stackrel{\text{def}}{=} \arg\max_p p \cdot (1 - F_i(p))$ . Call  $p_i$  the *optimal monopoly price*. Hartline and Roughgarden [18] study the auction  $VCG_{\mathbf{p}}$  that sets the optimal monopoly prices as reserve prices. They show that for single-item settings<sup>8</sup> and regular distributions  $F_1, \dots, F_n$ , the auction  $VCG_{\mathbf{p}}$  obtains a revenue of at least  $\frac{1}{2}$  of the optimal auction. We will show that our auction  $\mathcal{B}$  obtains a constant fraction of  $VCG_{\mathbf{p}}$ . Combining this with Hartline and Roughgarden's result, we can conclude that our auction  $\mathcal{B}$  obtains a constant fraction of the optimal revenue.

<sup>8</sup>Their proof applies to the more general matroid settings, which include single item settings as a special case.

We first prove that  $q_i \leq p_i$ . Since  $p_i$  is the optimal monopoly price, it satisfies  $(1 - F_i(p_i)) \cdot p_i \geq (1 - F_i(r)) \cdot r$  for any  $r \neq p_i$ . In particular, we can plug in  $r = \mu_i - \sigma_i k_i$ , player  $i$ 's reserve price in our digital auction  $\mathbb{A}$ . This gives us  $(1 - F_i(p_i)) \cdot p_i \geq (\mu_i - \sigma_i k_i) \cdot 1 - F_i(\mu_i - \sigma_i k_i)$ . Applying Cantelli's inequality and our results from lemma 2, we obtain

$$p_i \cdot (1 - F_i(p_i)) \geq (\mu_i - \sigma_i k_i) \cdot (1 - F_i(\mu_i - \sigma_i k_i)) \geq (\mu_i - \sigma_i k_i) \frac{k_i^2}{1 + k_i^2} = \mu_i - \frac{3}{2} \sigma_i k_i = q_i.$$

Since  $1 - F_i(p_i) < 1$ , we have  $p_i \geq (1 - F_i(p_i))p_i \geq q_i$ . This shows that our parametric reserve price  $q_i$  is always less than or equal to the optimal monopoly reserve price  $p_i$ .

Now we proceed to show that  $VCG_q$  is competitive with  $VCG_p$ . Separate the space of valuations into regions:

- $A_0 \stackrel{\text{def}}{=} \{(v_1, \dots, v_n) : \exists i_1, i_2 \text{ such that } v_{i_1} \geq p_{i_1}, v_{i_2} \geq p_{i_2}\}$ .
- $A_i \stackrel{\text{def}}{=} \{(v_1, \dots, v_n) : v_{-i} < p_{-i}, v_i \geq p_i\}$ .

Denote by  $Rev_{A_i}(VCG_p, \mathbf{F})$  the expected revenue that auction  $VCG_p$  obtains on the set  $A_i$ . The sets  $A_0, A_1, \dots, A_n$  are pairwise disjoint. Furthermore, notice that  $VCG_p$  obtains zero revenue outside the sets  $A_0, \dots, A_n$ . Thus, we can write

$$Rev(VCG_p, \mathbf{F}) = \sum_{i=0}^n Rev_{A_i}(VCG_p, \mathbf{F}).$$

Our strategy will be to prove that

1.  $VCG_q$  makes at least as much money as  $VCG_p$  on the set  $A_0$ .
2.  $\frac{Rev_{A_i}(VCG_q, \mathbf{F})}{Rev_{A_i}(VCG_p, \mathbf{F})} > \psi\left(\frac{\mu_i}{\sigma_i}\right)$  where  $\psi$  is a function of  $\frac{\mu_i}{\sigma_i}$  which we define below.
3. Show that the function  $\psi(x)$  is increasing when  $x > 0$ .

On the set  $A_0$ , our auction  $\mathcal{B} = VCG_q$  obtains at least as much revenue as  $VCG_p$ . Given a vector of valuations  $\mathbf{v} = (v_1, \dots, v_n)$  in  $A_0$ , let  $Q(\mathbf{v}) = \{i : v_i > q_i\}$  be the set of players whose valuation is above their parametric reserve price  $q_i$ . Analogously, let  $P(\mathbf{v}) = \{i : v_i > p_i\}$  be the set of players whose valuation is above their monopoly reserve price  $p_i$ . Since  $p_i \geq q_i$ , we have  $P(\mathbf{v}) \subset Q(\mathbf{v})$ . Since the vector  $\mathbf{v} \in A_0$ , the set  $P(\mathbf{v})$  contains at least two players. Thus,  $VCG_p$  sells to the player  $i^*$  in  $P(\mathbf{v})$  with the highest valuation and charges her the valuation of the second highest player  $j^*$  in  $P(\mathbf{v})$ . Both players  $i^*, j^*$  are in the set  $Q(\mathbf{v})$ . Thus, the auction  $VCG_q(\mathbf{v})$  obtains at least as much revenue as  $VCG_p(\mathbf{v})$  when  $\mathbf{v} \in A_0$ .

Now all we need to show is that the expected revenue that  $VCG_q$  obtains on set  $A_i$  is at least a constant fraction of the revenue that  $VCG_p$  obtains on the same set. On set  $A_i$ ,  $VCG_p$  will sell to player  $i$  at reserve price  $p_i$ . The auction's revenue on this set is exactly  $p_i$ . The auction  $VCG_q$  will also sell the good, since there exists at least one player (player  $i$ ) whose valuation  $v_i$  is above her reserve price. This is because  $v_i \geq p_i \geq q_i$ . The auction will sell the good for a price greater than or equal to  $q_i$  on this set, and thus obtains a revenue greater than or equal to  $q_i$ . Thus,  $\frac{Rev_{A_i}(VCG_q)}{Rev_{A_i}(VCG_p)} \geq \frac{q_i}{p_i}$ .

Now it suffices to give a lower bound on  $\frac{q_i}{p_i} \geq \psi\left(\frac{\mu_i}{\sigma_i}\right)$  for some increasing function  $\psi(\cdot)$ .<sup>9</sup> Recall that  $q_i$  is equal by definition to  $q_i(\mu_i, \sigma_i) = \mu_i - \frac{3}{2} \sigma_i k_i$ , where  $k_i$  is the unique real solution to the equation  $\frac{\mu_i}{\sigma_i} = \frac{1}{2}(k_i^3 + 3k_i)$ . To bound  $\frac{q_i}{p_i}$ , it suffices to give an upper bound on  $p_i$ . We do this by separating in two cases.

The first case is when  $p_i < \mu_i$ . In this case, we have  $\frac{q_i}{p_i} \geq 1 - \frac{3}{2} \frac{\sigma_i}{\mu_i} \cdot k_i$ . The expression  $1 - \frac{3}{2} \frac{\sigma_i}{\mu_i} \cdot k_i$  is the function  $\rho\left(\frac{\mu_i}{\sigma_i}\right)$  defined in section 4. We showed in theorem 2 that this function is increasing with  $\frac{\mu_i}{\sigma_i}$ .

<sup>9</sup>In section 7.1 we give an explicit constant lower bound for distributions with monotone hazard rate.

The second case is when  $p_i \geq \mu_i$ . Since  $p_i$  is the optimal monopoly reserve price, it must be the case that  $p_i \cdot (1 - F_i(p_i)) \geq r \cdot (1 - F(r))$  for any price  $r \neq p_i$ . In particular, we can choose  $r_i \stackrel{\text{def}}{=} \mu_i - \sigma_i k_i$ . As in our analysis above and in theorem 1, we have  $r_i \cdot (1 - F(r_i)) \geq \mu_i - \frac{3}{2}\sigma_i k_i = q_i$ . Thus, the optimal monopoly reserve price must satisfy  $p_i \cdot (1 - F_i(p_i)) \geq q_i$ .

Since  $p_i \geq \mu_i$ , we can write  $p_i = \mu_i + \sigma_i t$  for some non-negative  $t$ . Using a right-tailed version of Cantelli's inequality, we can give a bound

$$1 - F_i(\mu_i + \sigma_i t) \leq \frac{1}{1 + t^2}.$$

Combining this with the fact that  $p_i \cdot (1 - F_i(p_i)) \geq q_i$ , we have the inequality

$$(\mu_i + \sigma_i t) \cdot \frac{1}{1 + t^2} \geq q_i$$

$$\mu_i + \sigma_i t \geq q_i(1 + t^2)$$

$$q_i(1 + t^2) - \sigma_i t - \mu_i \leq 0.$$

Thus, in order for  $p_i = \mu_i + \sigma_i t$  to be the optimal monopoly reserve price, the above quadratic polynomial in  $t$  must be less than zero. Since the principal coefficient of this polynomial is  $q_i > 0$ , the polynomial is only negative between its roots. Thus, we must have that  $t$  lies in some interval  $[z_1, z_2]$  where  $z_1, z_2$  are the roots of  $q_i(1 + t^2) - \sigma_i t - \mu_i = 0$ , and we can write  $p_i = \mu_i + \sigma_i t \leq \mu_i + \sigma_i z_2$ .

We now have to write  $z_2$  in terms of  $\mu_i, \sigma_i$ . Recall that  $z_2$  is the largest root of the quadratic equation  $q_i(1 + t^2) - \sigma_i t - \mu_i = 0$ . Divide both sides of this equation by  $\mu_i$  to obtain

$$\frac{q_i}{\mu_i}(1 + t^2) - \frac{\sigma_i}{\mu_i}t - 1 = 0.$$

Since  $q_i = \mu_i - \frac{3}{2}\sigma_i k_i$ , we have that  $\frac{q_i}{\mu_i} = 1 - \frac{3}{2}\frac{\sigma_i}{\mu_i}k_i \stackrel{\text{def}}{=} \rho(\frac{\mu_i}{\sigma_i})$ , our function defined in section 4. Thus, we can write our equation as

$$\rho(\frac{\mu_i}{\sigma_i}) \cdot (1 + t^2) - \frac{\sigma_i}{\mu_i}t - 1 = 0.$$

For convenience of notation, write  $a \stackrel{\text{def}}{=} \frac{\mu_i}{\sigma_i}$ . We want to solve  $\rho \cdot (1 + t^2) - \frac{1}{a}t - 1 = 0$ . Using the quadratic formula, we get

$$\begin{aligned} z_2 &= \frac{1 + \sqrt{1 + 4a^2\rho - 4a^2\rho^2}}{2a\rho} \\ z_2 &= \frac{1}{2a\rho} + \sqrt{\frac{1}{4a^2\rho^2} + \frac{4a^2\rho}{4a^2\rho^2} - \frac{4a^2\rho^2}{4a^2\rho^2}} \\ z_2 &= \frac{1}{2a\rho} + \sqrt{\frac{1}{4a^2\rho^2} + \frac{1}{\rho} - 1}. \end{aligned}$$

Since  $\rho$  is a function of  $a = \frac{\mu_i}{\sigma_i}$ , the root  $z_2$  is also a function of  $a$ . Furthermore, the function  $z_2(a)$  is decreasing with both  $\rho$  and  $a$ . Since  $\rho$  is an increasing function of  $a$ , we conclude that  $z_2$  is decreasing in  $a = \frac{\mu_i}{\sigma_i}$ . We conclude that

$$p_i = \mu_i + \sigma_i t \leq \mu_i + \sigma_i z_2(\frac{\mu_i}{\sigma_i})$$

where  $z_2(\cdot)$  is a decreasing function. Combining this with the fact that  $\rho(\frac{\mu_i}{\sigma_i})$  is by definition equal to  $\frac{q_i}{\mu_i}$ , we get

$$\frac{q_i}{p_i} = \frac{\frac{q_i}{\mu_i}}{\frac{p_i}{\mu_i}} = \frac{\rho(\frac{\mu_i}{\sigma_i})}{1 + \frac{\sigma_i}{\mu_i}t} \geq \frac{\rho(\frac{\mu_i}{\sigma_i})}{1 + \frac{\sigma_i}{\mu_i} \cdot z_2(\frac{\mu_i}{\sigma_i})}.$$

Define

$$\psi(\frac{\mu_i}{\sigma_i}) = \frac{\rho(\frac{\mu_i}{\sigma_i})}{1 + \frac{\sigma_i}{\mu_i} \cdot z_2(\frac{\mu_i}{\sigma_i})}.$$

By our analysis above, the numerator is increasing in  $\frac{\mu_i}{\sigma_i}$  and the denominator is decreasing in  $\frac{\mu_i}{\sigma_i}$ . Thus  $\psi(\cdot)$  is an increasing function. Note also that  $\psi(\frac{\mu_i}{\sigma_i}) \leq \rho(\frac{\mu_i}{\sigma_i})$ , so  $\psi(\cdot)$  is also a lower bound on  $\frac{q_i}{p_i}$  for the case  $p_i \leq \mu_i$ .

Since  $\psi(\cdot)$  is increasing, and we assumed that the distributions  $F_1, \dots, F_n$  are in the class  $\mathcal{F}_c$ , we can infer that

$$\frac{Rev_{A_i}(VCG_{\mathbf{q}}, \mathbf{F})}{Rev_{A_i}(VCG_{\mathbf{p}}, \mathbf{F})} > \psi(c).$$

Since  $\frac{Rev_{A_0}(VCG_{\mathbf{q}}, \mathbf{F})}{Rev_{A_0}(VCG_{\mathbf{p}}, \mathbf{F})} \geq 1$ , this implies that

$$\frac{Rev(VCG_{\mathbf{q}}, \mathbf{F})}{Rev(VCG_{\mathbf{p}}, \mathbf{F})} \geq \psi(c).$$

Finally, applying Theorem 3.7 from Hartline and Roughgarden [18] we get

$$\frac{Rev(VCG_{\mathbf{p}}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq \frac{1}{2}.$$

Combining this with our result we obtain

$$\frac{Rev(VCG_{\mathbf{q}}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq \frac{1}{2} \cdot \psi(c).$$

Since  $\mathcal{B} = Rev(VCG_{\mathbf{q}}, \mathbf{F})$ , this proves our theorem. *Q.E.D.*

## 7.1 An explicit lower bound for distributions with monotone hazard rate.

In theorem 5, we gave a parametric auction with a competitive ratio bounded below by  $\frac{1}{2}\psi(c)$  for all distributions in  $\mathcal{F}_c$ . In this section, we give an explicit bound for all distributions with a monotone hazard rate.

**Theorem 6.** *When each  $F_i$  has a monotone hazard rate, the competitive ratio of our auction  $\mathcal{B}$  is bounded below by*

$$\frac{Rev(\mathcal{B}, \mathbf{F})}{Rev(OPT, \mathbf{F})} > 1.23\%.$$

*Proof.* Recall that a key step in the proof of theorem 5 was giving a bound  $\frac{q_i}{p_i} \geq \psi(\frac{\mu_i}{\sigma_i})$ , where  $q_i$  is our parametric reserve price and  $p_i$  is the monopoly reserve price that the optimal auction would set. This allowed us to conclude that  $\frac{Rev(\mathcal{B}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq \frac{1}{2}\psi(\frac{\mu_i}{\sigma_i})$ . By improving the bound on  $\frac{q_i}{p_i}$ , we can improve the competitive ratio of our auction.

In this proof, we show that for  $F_i$  with monotone hazard rate, the ratio  $\frac{q_i}{p_i}$  satisfies the bound

$$\frac{q_i}{p_i} \geq 2.47\%.$$

Once we show this, the reasoning from theorem 5 will allow us to conclude that

$$\frac{Rev(\mathcal{B}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq \frac{1}{2} \cdot 2.47\% \geq 1.23\%.$$

We use two facts about distributions with a monotone hazard rate

1. When  $F_i$  has a monotone hazard rate,  $\frac{\mu_i}{\sigma_i} \geq 1$ .<sup>10</sup> Thus, we have  $F_i \in \mathcal{F}_1$  for each player  $i$ .

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<sup>10</sup>Barlow, Marshall and Proschan [4] noted that for a random variable  $X$  with monotone hazard rate,  $E[X^2] \leq 2E[X]^2$ . Since  $Var(X) = E[X^2] - E[X]^2$ , this gives us  $Var(X) \leq E[X]^2$ . Taking square roots, we obtain  $\sigma \leq \mu$ .



2. For any distribution  $F_i$  with a monotone hazard rate and with mean  $\mu_i$ , we have  $1 - F_i(\mu_i) \geq \frac{1}{e}$ . [3]

We now give a bound on  $\frac{q_i}{p_i}$ . Recall that  $q_i$  is equal by definition to  $q_i(\mu_i, \sigma_i) = \mu_i - \frac{3}{2}\sigma_i k_i$ , where  $k_i$  is the unique real solution to the equation  $\frac{\mu_i}{\sigma_i} = \frac{1}{2}(k^3 + 3k)$ . We now give a bound on  $p_i$  by separating into two cases.

We begin with the case where  $p_i \leq \mu_i$ . In this case, as shown in theorem 5, we have  $\frac{q_i}{p_i} \geq 1 - \frac{3}{2}\sigma_i k_i \stackrel{\text{def}}{=} \rho(\frac{\mu_i}{\sigma_i})$ . We showed in theorem 2 that  $\rho(\cdot)$  is an increasing function. Since  $F_i$  has a monotone hazard rate, the observation above tells us that  $\frac{\mu_i}{\sigma_i} \geq 1$ . Thus,

$$\frac{q_i}{p_i} \geq \rho(1) \geq 10.5\%.$$

The complicated case is when  $p_i \geq \mu_i$ . Since  $p_i$  is the optimal monopoly reserve price, it satisfies  $p_i \cdot (1 - F_i(p_i)) \geq r \cdot (1 - F_i(r))$  for every  $r \neq p_i$ . In particular, we can choose  $r = \mu_i$  and obtain  $p_i \cdot (1 - F_i(p_i)) \geq \mu_i \cdot (1 - F_i(\mu_i)) \geq \frac{\mu_i}{e}$ .

Write  $p_i = \mu_i + \sigma_i t$  for some  $t > 0$ . We will give an upper bound on  $t$ . A right-tailed version of Cantelli's inequality tells us that

$$1 - F_i(p_i) = 1 - F_i(\mu_i + \sigma_i t) \leq \frac{1}{1 + t^2}.$$

Combining this with the fact that  $p_i \cdot (1 - F_i(p_i)) \geq \frac{\mu_i}{e}$ , we get

$$\begin{aligned} (\mu_i + \sigma_i t) \frac{1}{1 + t^2} &\geq \frac{\mu_i}{e}. \\ (1 + \frac{\sigma_i}{\mu_i} t) &\geq \frac{1 + t^2}{e}. \end{aligned}$$

Since  $F_i$  has a monotone hazard rate,  $\frac{\sigma_i}{\mu_i} \leq 1$ . This yields  $1 + t \geq 1 + \frac{\sigma_i}{\mu_i} t$ . Combining this with our above inequality, we get

$$1 + t \geq \frac{1 + t^2}{e}.$$

A calculation shows that  $t$  satisfies this inequality only if  $t \leq \frac{e}{2} + \frac{1}{2}\sqrt{-4 + 4e + e^2} \leq 3.25$ . Thus, we have  $p_i \leq \mu_i + 3.25\sigma_i \leq 4.25\mu_i$ , where the last inequality is derived by using the fact that  $\mu_i \geq \sigma_i$ .

We can now give the following bound on  $\frac{q_i}{p_i}$ :

$$\frac{q_i}{p_i} \geq \frac{\mu_i - \frac{3}{2}\sigma_i k_i}{4.25\mu_i} = \frac{1 - \frac{3}{2}\frac{\sigma_i}{\mu_i} k_i}{4.25} = \frac{\rho(\frac{\mu_i}{\sigma_i})}{4.25} \geq \frac{\rho(1)}{4.25}$$

where the last inequality is because  $\rho(\cdot)$  is an increasing function and  $\frac{\mu_i}{\sigma_i} \geq 1$ . Since  $\rho(1) \geq 10.5\%$ , we obtain  $\frac{q_i}{p_i} \geq 2.47\%$ . Using the reasoning from theorem 5, we conclude that

$$\frac{Rev(\mathcal{B}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq 1.23\%.$$

*Q.E.D.*

## 8 Conclusion and Future Work

We introduced parametric auctions, a new type of auction where the seller only needs to know the parameters  $\mu_i, \sigma_i$  of the buyer distributions  $F_1, \dots, F_n$ . For the single-item and digital goods settings, we constructed auctions that obtained a constant fraction of the optimal auction revenue for a large class of distributions, including all monotone hazard rate distributions. Furthermore, our digital auction  $\mathbb{A}$  is the best parametric

posted price mechanism in a maximin sense. When an adversary chooses the distribution  $\mathbf{F}$ , our auction  $\mathbf{A}$  maximizes the worst-case revenue.

Our results make extensive use of moment bounds both for general distributions and for monotone hazard rate distributions. The theory of moment bounds is very well developed, but to the best of our knowledge has never been applied to auctions before. In the future, we are interested in analyzing parametric auctions with multi-dimensional types, extending our work to other settings such as downward closed environments and public goods, and evaluating empirically the performance of our parametric auctions. We believe that our model and techniques are very robust, and that they can be widely applied.

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